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A LOST PROOF RELATED TO TRANSCENDENTAL FUNCTIONS

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Abstract

The purpose of this paper is to give a proof for a well-known result, which characterizes transcendental entire functions.

1. Introduction

By a *transcendental* function, we mean a function f, such that for any nonzero polynomial $P \in \mathbb{C}[X, Y]$, the function $P(z, f(z))$ is not the zero function, otherwise f is said to be *algebraic*. A helpful result on transcendental functions asserts that, if f is an *entire* function, namely, a function which is analytic in $\mathbb{C}$, to say that f is a transcendental function amounts to say that it is not a polynomial. Maybe this result is one of the most quoted when we wish to construct a transcendental function, because it allows us to decide when a given function is or not transcendental, only by looking at its Taylor series. In the last year, the professor Waldschmidt pointed me about this result. In that occasion, we used it, however, we did not find a reference with its proof. Surely,

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somebody proved it, but where is this proof ? Stories like that are more common in mathematics than we imagine. Hence, the goal of this work is to give an effective proof for this powerful fact, which can be cited by mathematicians who are uncomfortable with this kind of folklore result. Let us state it for the sake of preciseness.

Proposition 1. *An entire function is algebraic, if and only if it is a polynomial.*

As an immediate consequence, it follows the transcendence of the omnipresent exponential function $\exp(z) = \sum_{n=0}^{\infty} z^n / n!$. More generally, the series $\sum_{n=0}^{\infty} a_n z^n$ gives a transcendental entire function, when it converges absolutely and uniformly on any compact subsets of $\mathbb{C}$ and $\{n \in \mathbb{N} : a_n \neq 0\}$ is an infinite set. Here, $\mathbb{N}$ denotes the set of natural numbers.

This result is very useful for generating transcendental functions of an effective way. For instance, when we work with interpolation series of the form $f(z) := \sum_{n=0}^{\infty} a_n P_n(z)$, for which we want to have a couple of desired properties, where one of them is f to be transcendental, then we must be only worried in considering polynomials P_n, whose high order derivates vanish at some fixed points. After, it is enough to take a_n nonzero complex numbers.

2. Auxiliaries Lemmas and the Lost Proof

Let f be an entire function and $R > 0$. We set $|f|_R = \sup_{|z|=R} |f(z)|$. Before going to the proof of the Proposition 1, we need a couple of technical lemmas.

Lemma 1. *If f is a non-constant entire function, then there exists a real number $R_0 > 0$, such that $|f|_R \geq 1$ for all $R \geq R_0$.*

Proof. Supposing the contrary, we have that for any $z \in \mathbb{C}$, there exists $R > |z|$ such that $|f|_R < 1$. Since that

$$|f|_R = \sup_{|z|=R} |f(z)| = \sup_{|z|\leq R} |f(z)|,$$

then $|f(z)| < 1$. But z is an arbitrary complex number, it follows that f is bounded and then a constant function, by Liouville's theorem. □

Lemma 2. *Let $P(z) \in \mathbb{C}[z]$ be a nonzero polynomial. Then there exist $k > 0$ and $R_1 > 0$, such that $|P(z)| > k$, for all $z \in \mathbb{C}$ with $|z| \geq R_1$.*

Proof. When P is a nonzero constant polynomial, the result follows trivially. Let P be a non-constant polynomial and suppose that for $k_n = \frac{1}{n}$ and $R_{1n} = n$, there exists $z_n \in \mathbb{C}$, with $|z_n| > R_{1n} = n$, and such that $|P(z_n)| \leq k_n = \frac{1}{n}$. Hence, when $n \to \infty$, we have that $|z_n|$ tends to the infinity and $|P(z_n)|$ tends to zero. Contradiction, because it being P non-constant, $\lim_{|z|\to\infty} |P(z)| = \infty$. □

Lemma 3. *If f is an algebraic entire function, then there exist $c > 0$ and $R_2 > 0$, such that $|f|_R \leq R^c$ for all $R \geq R_2$.*

Proof. Set $P(x, y) \in \mathbb{C}[x, y]$, nonzero, satisfying $P(z, f(z)) = 0$, for all $z \in \mathbb{C}$. Surely, we can rewrite the previous equality as

$$a_0(z) + a_1(z)f(z) + \cdots + a_d(z)f(z)^d = 0, \text{ for all } z \in \mathbb{C}, \tag{1}$$

where $a_0(z), \ldots, a_d(z) \in \mathbb{C}[z]$ and $a_d(z)$ is a nonzero polynomial. From Lemma 2, there exist positive real numbers k and R_1, such that $|a_d(z)| > k$ for all z with $|z| \geq R_1$. From Lemma 1, there exists $R_0 > 0$ such that $|f|_R \geq 1$, for $R \geq R_0$. From now on, all R that to appear will be considered $\geq \max\{R_0, R_1\}$. Since $|f(z)|$ is a continuous function and

$\partial B(0;\, R)$[1] is a compact set, there exists $z' \in \mathbb{C}$, with $|z'| = R$ and such that $|f(z')| = \sup\limits_{|z|=R} |f(z)| = |f|_R$. Replacing z by z' in the equality (1), we obtain

$$\underbrace{a_0(z')}_{z_1} + \underbrace{a_1(z')f(z') + \cdots + a_d(z')f(z')^d}_{z_2} = 0.$$

Since $|z_2| - |z_1| \le |z_1 + z_2| = 0$. We have $|z_2| \le |z_1|$, that is,

$$|a_1(z')f(z') + \cdots + a_d(z')f(z')^d| \le |a_0(z')|.$$

Using that $|f(z')| \ge 1$, it follows that $|a_1(z') + \cdots + a_d(z')f(z')^{d-1}| \le |a_0(z')|$.

Repeating the same process $d - 1$ times, we get

$$|a_d(z')f(z')| \le |a_0(z')| + \cdots + |a_{d-1}(z')|.$$

Since that $|z'| = R \ge R_1$, then $|a_d(z')| > k$. So

$$|f(z')| \le \frac{1}{k}(|a_0(z')| + \cdots + |a_{d-1}(z')|) \le |b_0| + \cdots + |b_s||z'|^s,$$

where $b_0, \ldots, b_s \in \mathbb{C}$ and $s = \max\{\deg a_0(z), \ldots, \deg a_{d-1}(z)\}$. Again, we use that $|z'| = R$ and $|f(z')| = |f|_R$ for obtaining

$$|f|_R \le |b_0| + \cdots + |b_s|R^s.$$

There exists a sufficient large $R_2 > \max\{R_0, R_1\}$ such that $\dfrac{R^{s+1}}{|b_0| + \cdots + |b_s|R^s} \ge 1$. Therefore, for $R \ge R_2$, we have $|f|_R \le R^{s+1}$. □

Now, we are able to prove our main result.

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[1] the boundary of the ball centered in the origin and with radius R.

Proof (Proposition 1)**.** When $f(z) \in \mathbb{C}[z]$, we define $P(x, y) = f(x) - y$, so $P(z, f(z)) = 0$, for all $z \in \mathbb{C}$. For the necessary condition, given $n \geq c + 1$, by Cauchy's integral formula, for $R \geq R_2$ (where c and R_2 are those obtained in Lemma 3), we have

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi} \int_{|w|=R} \frac{|f(w)|}{|w|^{n+1}} dw \overset{|w|=R}{\leq} \frac{n!}{2\pi R^{n+1}} \int_{|w|=R} |f|_R dw \overset{\text{Lemma 3}}{\leq} \frac{n!}{R^{n-c}}.$$

Since $n - c \geq 1$, when R tends to the infinity, we get $|f^{(n)}(0)| = 0$. Thus, $f^{(k)}(0) = 0$ for all $k \geq c + 1$. But, f is an entire function and hence its Laurent series reduces to

$$f(z) = f(0) + \cdots + \frac{f^{(c)}(0)}{c!} z^c + \left(\sum_{k \geq c+1} \frac{f^{(k)}(0)}{k!} z^k \right).$$

Therefore, f is a polynomial with degree $\leq c$. $\square$

Remark 1. Note that the function $\frac{1 + z}{z^2 + 1}$ is algebraic, but it is not a polynomial. That does not contradict the Proposition 1, because f is not an entire function (it has poles at $z = \pm\sqrt{-1}$).

The results used in this work about complex variables (Cauchy's integral formula, maximum modulus theorem and Liouville's theorem) can be found in any good book about the subject. As an example, see [1].

References

[1] J. B. Conway, Functions of a Complex Variable, Springer, New York, 1973.

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